TSRT14: Sensor Fusion
Lecture 3

— Cramér-Rao lower bound (CRLB)
— Sensor networks

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Le 3: Sensor networks and detection theory

Whiteboard:

- Selected derivations.

Slides:

- Typical sensor networks measurements
- Nonlinear models in sensor networks
- Dedicated LS solutions.

Summary Lecture 2

Nonlinear model: $\mathbf{y} = \mathbf{h}(x) + \mathbf{e}$, $\operatorname{cov}(\mathbf{e}) = \mathbf{R}$.
NWLS minimizes

$$\hat{x}^{\text{NWLS}} = \arg\min_x V^{\text{NWLS}}(x) = \arg\min_x \frac{1}{2}\big(\mathbf{y} - \mathbf{h}(x)\big)^T \mathbf{R}^{-1}\big(\mathbf{y} - \mathbf{h}(x)\big).$$

NLT: Approximate $z = g(u)$, $u \sim \mathcal{N}(\hat{u}, P_u)$ with $z \sim \mathcal{N}(\hat{z}, P_z)$.
Variations: TT1, TT2, UT or MCT.

- The *direct approach*, where $x = \mathbf{h}^{-1}(\mathbf{y} - \mathbf{e})$ is approximated.
- The *indirect approach*, where the distribution of $\mathbf{y} = \mathbf{h}(x)$ is approximated using a prior of $x \sim \mathcal{N}(\hat{x}, P^{xx})$: The trick is to consider the mapping

$$u = \begin{pmatrix} x \\ e \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \bar{x} \\ 0 \end{pmatrix}, \begin{pmatrix} P^{xx} & 0 \\ 0 & R \end{pmatrix}\right)$$

$$z = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ h(x,e) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \begin{pmatrix} P^{xx} & P^{xy} \\ P^{yx} & P^{yy} \end{pmatrix}\right)$$

and then apply

$$\hat{x} = \bar{x} + P^{xy}\big(P^{yy}\big)^{-1}\big(y - \bar{y}\big),$$

Cramér-Rao Lower Bound (CRLB)

Cramér-Rao Lower Bound (CRLB)

Theorem (Cramér-Rao Lower Bound)

For any unbiased $\hat{x}$ ($\mathsf{E}(\hat{x}) = x^0$), the estimate satisfies

$$\text{cov}(\hat{x}) \succeq \mathcal{I}^{-1}(x^0),$$

where $\mathcal{I}(x)$ is the Fisher information matrix (FIM),

$$\begin{aligned}\mathcal{I}(x) &= -\,\mathsf{E}\Big(\nabla_x^2 \log p_{\mathbf{e}}\big(\mathbf{y} - \mathbf{h}(x)\big)\Big)\\ &= \mathsf{E}\Big\{\nabla_x^T\Big(\log p_{\mathbf{e}}\big(\mathbf{y} - \mathbf{h}(x)\big)\Big)\,\nabla_x\Big(\log p_{\mathbf{e}}\big(\mathbf{y} - \mathbf{h}(x)\big)\Big)\Big\}\end{aligned}$$

under mild regularity conditions, see Appendix C.

Cramér-Rao Lower Bound (CRLB): theorems

Theorems:

- $\hat{x}^{\text{ML}}$ is efficient: asymptotically $\mathsf{E}(\hat{x}) = x^0$ and $\text{cov}(\hat{x}) = \mathcal{I}^{-1}(x^0)$.
- If $\mathbf{h}(x) = \mathbf{H}x$, then:
  - $\hat{x}^{\text{WLS}}$ is a *best linear unbiased estimator* (BLUE).
  - $\text{cov}(\hat{x}^{\text{WLS}}) = (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1} \succeq \mathcal{I}^{-1}(x^0)$ with equality if and only if $\mathbf{e}$ is Gaussian.
- If $p_{\mathbf{e}}(x) = \mathcal{N}(\mathbf{e}; \mathbf{0}, \mathbf{R})$, then
$$\mathcal{I}(x) = \big(\nabla_x^T\,\mathbf{h}(x)\big)\mathbf{R}^{-1}\big(\nabla_x\,\mathbf{h}(x)\big).$$

Sensor Networks

Chapter 4 Overview

- Sensor model $y_k = h_k(x) + e_k$ as before.
- Basic sensor models, specific forms of $h(x)$.
- Geometrical interpretation (TDOA).
- Overview of algorithms with examples.
- Tricks: transform to linear model or conditionally linear model.

Sensor Models in Radio Networks

Received signal $y_k(t)$ as a noisy, delayed and attenuated version of the transmitted signal $s(t)$

$$y_k(t) = a_k s(t - \tau_k) + e_k(t), \quad k = 1, 2, \ldots, N.$$

Time-delay estimation using known training signal (pilot symbols) gives

$$r_k = \tau_k v = \|x - p_k\|_2 = \sqrt{(x_1 - p_{k,1})^2 + (x_2 - p_{k,2})^2}.$$

- *Time-of-arrival* (TOA) — transport delay.
- *Time-difference-of-arrival* (TDOA) — arrival time known, but the broadcast time not.
- Estimation of a_k gives *received signal strength* (RSS), which does not require known training signal, just transmitter power P_0 and path propagation constant α

$$P_k = P_0 - \alpha \log(\|x - p_k\|).$$

Basic Network Sensor Models

The basic network measurements in any network (radio, acoustic, sonar, seismic) are summarized as follows:

TOA $h_k(x) = r_k = \|x - p_k\|$

TDOA $h_k(x) = r_k = \|x - p_k\| + r_0$

DOA $h_k(x) = \varphi_k = \operatorname{arctan2}\big(x_2 - p_{k,2}, x_1 - p_{k,1}\big)$

RSS $h_k(x) = P^0 - \beta \log\big(\|x - p_k\|\big)$

Note:

All models are on the form $y_k = h_k(x) + e_k$.

TDOA Geometry

Common offset r_0 (due to unsynchronized clocks)

$$r_k = \|x - p_k\| + r_0, \quad k = 1, 2, \ldots, N.$$

Estimation approach:
Consider r_0 as a parameter (*cf.* GPS).

Common in literature:
Study range differences

$$r_{i,j} = r_i - r_j, \quad 1 \le i < j \le N.$$

Gives nice geometric interpretation.

TDOA: maths

Assume $p_1 = (D/2, 0)^T$ and $p_2 = (-D/2, 0)^T$, respectively, then

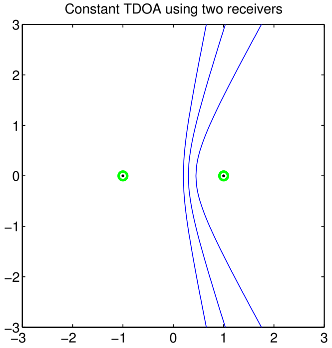
$$\begin{aligned} r_1 &= \sqrt{x_2^2 + (x_1 - D/2)^2} \\ r_2 &= \sqrt{x_2^2 + (x_1 + D/2)^2} \\ r_{12} &= r_2 - r_1 = h(x, D) \\ &= \sqrt{x_2^2 + (x_1 + D/2)^2} - \sqrt{x_2^2 + (x_1 - D/2)^2}. \end{aligned}$$

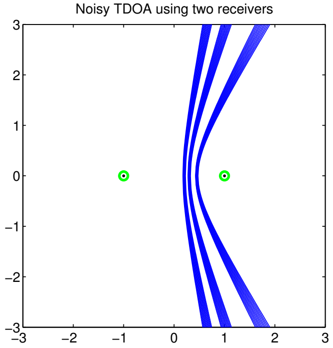
Simplify

$$\frac{x_1^2}{a} - \frac{x_2^2}{b} = \frac{x_1^2}{r_{12}^2/4} - \frac{x_2^2}{D^2/4 - r_{12}^2/4} = 1.$$

TDOA: illustration

Illustration of three different values of r_{12}.
Corresponds to three different hyperbolic functions.
Noise on r_{12} gives confidence bands.





Direction/Angle of Arrival (DOA/AOA)

The solution to this hyperbolic equation has asymptotes along the lines

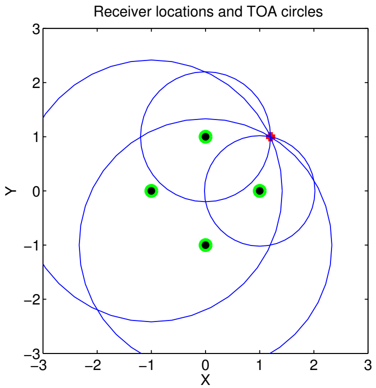
$$x_2 = \pm \frac{b}{a} x_1 = \pm \sqrt{\frac{D^2/4 - r_{12}^2/4}{r_{12}^2/4}} x_1 = \pm x_1 \sqrt{\left(\frac{D}{r_{12}}\right)^2 - 1}.$$

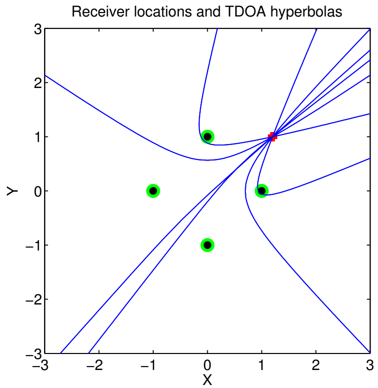
AOA, φ, for far-away transmitters (the far-field assumptions of planar waves):

$$\varphi = \pm \arctan\left(\sqrt{\left(\tfrac{D}{r_{12}}\right)^2 - 1}\right).$$

THE Example

Noise-free nonlinear relations for TOA and TDOA.





Estimation Criteria

General problem formulation and solution:

$$\hat{x} = \arg\min_x V(x).$$

Specific cases summarized

NLS: $V^{NLS}(x) = \|\mathbf{y} - \mathbf{h}(x)\|^2 = \big(\mathbf{y} - \mathbf{h}(x)\big)^T \big(\mathbf{y} - \mathbf{h}(x)\big)$

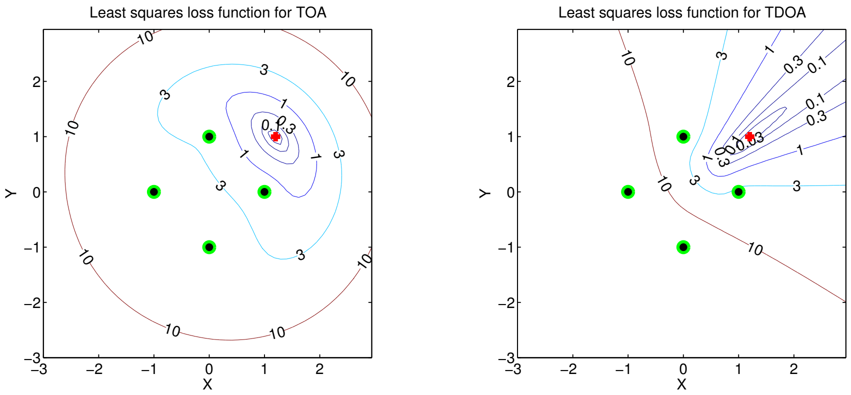
NWLS: $V^{NWLS}(x) = \big(\mathbf{y} - \mathbf{h}(x)\big)^T \mathbf{R}^{-1}(x) \big(\mathbf{y} - \mathbf{h}(x)\big)$

ML: $V^{ML}(x) = \log p_{\mathbf{e}}\big(\mathbf{y} - \mathbf{h}(x)\big)$

GLS: $V^{GML}(x) = \big(\mathbf{y} - \mathbf{h}(x)\big)^T \mathbf{R}^{-1}(x) \big(\mathbf{y} - \mathbf{h}(x)\big) + \log\det \mathbf{R}(x)$

THE Example

Level curves $V(x)$ for TOA and TDOA.



Estimation Methods

General principles:

Steepest descent

$$\hat{x}_k = \hat{x}_{k-1} + \mu_k \mathbf{H}^T(\hat{x}_{k-1}) R^{-1}(y - \mathbf{h}(\hat{x}_{k-1}))$$

Gauss-Newton

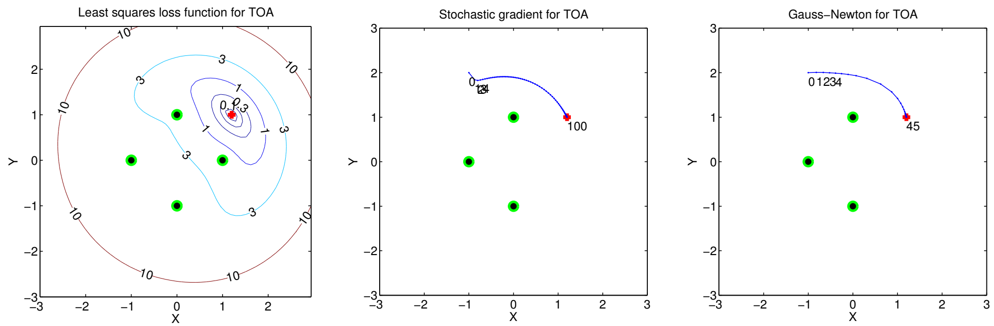
$$\hat{x}_k = \hat{x}_{k-1} + \mu_k \big(\mathbf{H}^T(\hat{x}_{k-1}) R^{-1} \mathbf{H}(\hat{x}_{k-1})\big)^{-1} \mathbf{H}^T(\hat{x}_{k-1}) R^{-1} \big(y - \mathbf{h}(\hat{x}_{k-1})\big)$$

Problem specific quantities:

Method	$h(x, p_i)$	$\partial h / \partial x_1$	$\partial h / \partial x_2$
RSS	$P_0 + 10\beta \log_{10} r_i$	$\frac{10\beta}{\log 10} \frac{x_1 - p_{i,1}}{r_i^2}$	$\frac{10\beta}{\log 10} \frac{x_2 - p_{i,2}}{r_i^2}$
TOA	r_i	$\frac{x_1 - p_{i,1}}{r_i}$	$\frac{x_2 - p_{i,2}}{r_i}$
TDOA	$r_i - r_j$	$\frac{x_1 - p_{i,1}}{D_i} - \frac{x_1 - p_{j,1}}{D_j}$	$\frac{x_2 - p_{i,2}}{D_i} - \frac{x_2 - p_{j,2}}{D_j}$
AOA	$\alpha_i + \arctan \frac{x_2 - p_{i,2}}{x_1 - p_{i,1}}$	$\frac{-(x_2 - p_{i,2})}{r_i^2}$	$\frac{x_1 - p_{i,1}}{r_i^2}$

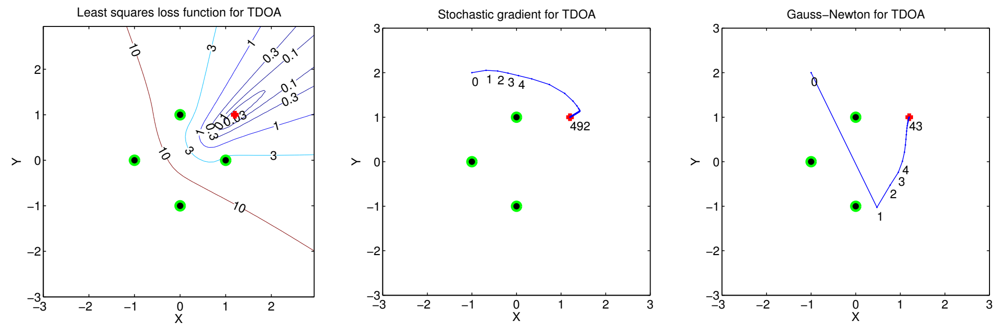
THE Example (1/3)

The steepest descent and Gauss-Newton algorithms for TOA.



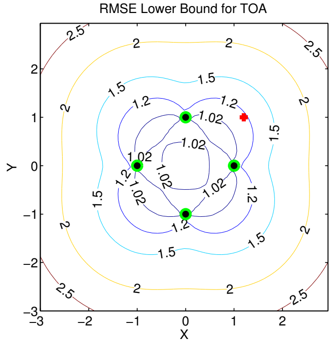
THE Example (2/3)

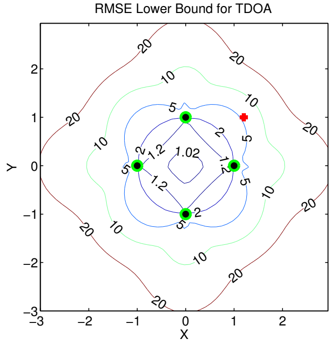
The steepest descent and Gauss-Newton algorithms for TDOA.



THE Example (3/3)

CRLB for TOA and TDOA: RMSE $(\hat{x}) \geq \sqrt{\operatorname{tr}(\mathcal{I}^{-1}(x^0))}$, where $\mathcal{I}(x^0)$ is the *Fisher information matrix* (FIM) evaluated at the true parameter x^0.





Dedicated Explicit LS Solutions

Basic trick: study NLS of *squared* distance measurements:

$$\hat{x} = \arg\min_x \sum_{k=1}^{N} \big(r_k^2 - \|x - p_k\|^2\big)^2.$$

Note:

What does this imply for the measurement noise? Several *ad hoc* solutions exist for transforming the nonlinear problem into a linear problem.

DOA Triangulation

Angle observations from sensor at position p_k

$$\varphi_k = \arctan\Big(\frac{x_2 - p_{k,2}}{x_1 - p_{k,1}}\Big)$$

$$(x_1 - p_{k,1})\tan(\varphi_k) = x_2 - p_{k,2}.$$

Linear model follows immediately,

$$\mathbf{y} = \mathbf{H}x + \mathbf{e}$$

$$\mathbf{y} = \begin{pmatrix} p_{1,1}\tan(\varphi_1) - p_{1,2} \\ p_{2,1}\tan(\varphi_2) - p_{2,2} \\ \vdots \\ p_{N,1}\tan(\varphi_N) - p_{N,2} \end{pmatrix}, \qquad \mathbf{H} = \begin{pmatrix} \tan(\varphi_1) & -1 \\ \tan(\varphi_2) & -1 \\ \vdots & \\ \tan(\varphi_N) & -1 \end{pmatrix}.$$

RSS (1/2)

Received signal strength (RSS) observations:

- All waves (radio, radar, IR, seismic, acoustic, magnetic) decay exponentially in range.
- Receiver k measures energy/power/signal strength for wave i:

$$P_{k,i} = P_{0,i}\|x - p_k\|^{\beta_i}.$$

- Transmitted signal strength and path loss constant unknown.
- Communication constraints make coherent detection from the signal waveform impossible.
- Compare $P_{k,i}$ for different receivers.

RSS (2/2)

Log model:

$$\bar{P}_{k,i} = \bar{P}_{0,i} + \beta_i \underbrace{\log(\|x - p_k\|)}_{=:c_k(x)}$$
$$y_{k,i} = \bar{P}_{k,i} + e_{k,i}$$

Use separable least squares to eliminate path loss constant and transmitted power for wave i:

$$\widehat{(x,\theta)} = \arg\min_{x,\theta} V(x,\theta)$$
$$V(x,\theta) = \sum_{i=1}^{M}\sum_{k=1}^{N} \frac{\big(y_{k,i} - h(c_k(x),\theta_i)\big)^2}{\sigma^2_{P,i}}$$
$$h(c_k(x),\theta_i) = \theta_{i,1} + \theta_{i,2} c_k(x)$$
$$c_k(x) = \log(\|x - p_k\|)$$

Finally, use NLS to optimize over 2D target position x.

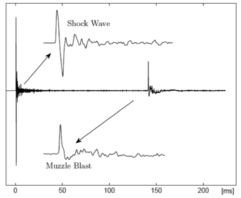
Shooter Localization (1/2)

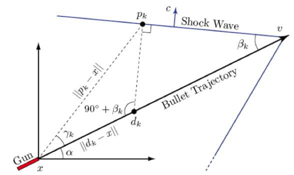
Network with 10 microphones around a 'camp'. Shooter aiming inside the camp. Supersonic bullet injects a shock wave followed by an acoustic muzzle blast. Fusion of time differences give shooter position and aiming point.

Model, with x being the position of the shooter,

$$y_k^{\text{MB}} = t_0 + b_k + \tfrac{1}{c}\|p_k - x\| + e_k^{\text{MB}},$$
$$y_k^{\text{SW}} = t_0 + b_k + \tfrac{1}{r}\log\frac{v_0}{v_0 - r\|d_k - x\|} + \tfrac{1}{c}\|d_k - p_k\| + e_k^{\text{SW}}.$$

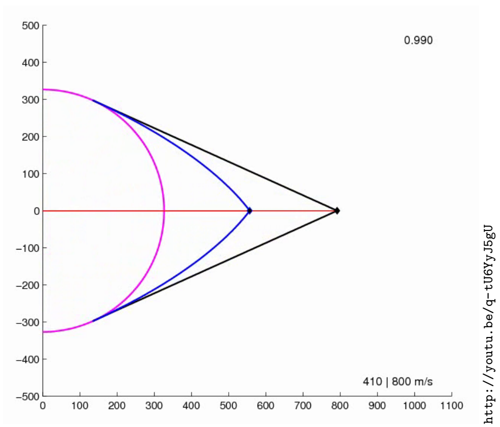
where d_k is an implicit function of the state.





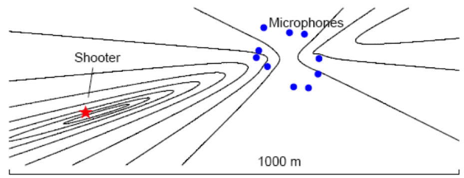
Shooter Localization (2/2)

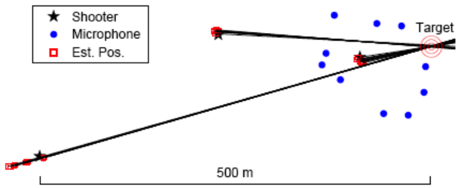
- Red: Bullet trajectory
- Black: Ideal shock wave
- Blue: Shock wave considering bullet retardation
- Magenta: Muzzle blast



Shooter Localization: results

- WLS loss function illustrates the information in $y_k^{\text{MB}} - y_k^{\text{SW}}$, $k = 1, 2, \ldots, N = 10$.
- Both shooter position and aiming direction α are well estimated for each shot.
- Also bullet's muzzle speed and ammunition length can be estimated!





Summary

Summary Lecture 3 (1/2)

- Signal model for localization in sensor networks

$$\mathbf{y} = \mathbf{h}(x - \mathbf{p}) + \mathbf{e}, \quad \text{cov}(\mathbf{e}) = \mathbf{R}$$

  x is the unknown position, $\mathbf{p}$ is the (known in this chapter) sensor locations.
- CRLB Theorem: for any unbiased estimator $\hat{x}$,

$$\text{cov}(\hat{x}) \succeq \mathcal{I}^{-1}(x^0),$$

  Fisher information matrix (FIM), $\mathcal{I}^{-1}(x)$.
- ML estimate *efficient*: asymptotically *unbiased* satisfying CRLB.

Summary Lecture 3 (2/2)

- The basic network measurements:
  - **TOA** $r_k = \|x - p_k\| + e_k$
  - **TDOA** $r_k = \|x - p_k\| + r_0 + e_k$
  - **DOA** $\varphi_k = \arctan2(x_2 - p_{k,2}, x_1 - p_{k,1}) + e_k$
  - **RSS** $y_k = P_0 - \beta \log(\|x - p_k\|)$

  NLS or NLT general approaches to estimate x.
- Tricks (not statistically optimal!)
  - **TOA** range parameter trilateration: r_k^2 is linear in $(x_1, x_2, x_1^2 + x_2^2)^T$
  - **TOA** reference sensor trilateration: $r_k^2 - r_1^2$ is linear in x
  - **DOA** triangulation approach: x is an affine function in $\tan(\varphi_k)$