TSRT14: Sensor Fusion
Lecture 6

— Kalman filter (KF)
— KF approximations (EKF, UKF)

Gustaf Hendeby

gustaf.hendeby@liu.se

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Le 6: Kalman filter (KF), approximations (EKF, UKF)

Whiteboard:

- Derivation framework for KF, EKF, UKF

Slides:

- Kalman filter summary: main equations, robustness, sensitivity, divergence monitoring, user aspects.
- Nonlinear transforms revisited.
- Application to derivation of EKF and UKF.
- User guidelines and interpretations.

Lecture 5: summary

- Standard models in global coordinates:
  - Translation $p_t^{(m)} = w_t^p$
  - 2D orientation for heading $h_t^{(m)} = w_t^h$
  - Coordinated turn model

$$\dot{X} = v^X \qquad \dot{Y} = v^Y$$
$$\dot{v}^X = -\omega v^Y \qquad \dot{v}^Y = \omega v^X$$
$$\dot{\omega} = 0$$

- Standard models in local coordinates (x, y, ψ)
  - Odometry and dead reckoning for (x, y, ψ)

$$X_t = X_0 + \int_0^t v_t^x \cos(\psi_t)\, dt \qquad Y_t = Y_0 + \int_0^t v_t^x \sin(\psi_t)\, dt$$
$$\psi_t = \psi_0 + \int_0^t \dot{\psi}_t\, dt$$

  - Force models for $(\dot{\psi}, a_y, a_x, \dots)$
  - 3D orientation $\dot{q} = \frac{1}{2} S(\omega) q = \frac{1}{2} \bar{S}(q) \omega$

Kalman Filter (KF)

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Chapter 7 Overview

Kalman filter

- Algorithms and derivation
- Practical issues
- Computational aspects
- Filter monitoring

The discussion and conclusions do usually apply to all nonlinear filters, though it is more concrete in the linear Gaussian case.

Kalman Filter (KF)

Time-varying state space model:

$$x_{k+1} = F_k x_k + G_k v_k, \qquad \text{cov}(v_k) = Q_k$$
$$y_k = H_k x_k + e_k, \qquad \text{cov}(e_k) = R_k$$

Time update:

$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k}$$
$$P_{k+1|k} = F_k P_{k|k} F_k^T + G_k Q_k G_k^T$$

Measurement update:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1} (y_k - H_k \hat{x}_{k|k-1})$$
$$P_{k|k} = P_{k|k-1} - P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1} H_k P_{k|k-1}.$$

KF Modifications

Auxiliary quantities: innovation ε_k, innovation covariance S_k and Kalman gain K_k

$$
\begin{aligned}
\hat{y}_k &= H_k \hat{x}_{k|k-1} \\
\varepsilon_k &= y_k - H_k \hat{x}_{k|k-1} = y_k - \hat{y}_k \\
S_k &= H_k P_{k|k-1} H_k^T + R_k \\
K_k &= P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1} = P_{k|k-1} H_k^T S_k^{-1}
\end{aligned}
$$

Filter form

$$
\begin{aligned}
\hat{x}_{k|k} &= F_{k-1}\hat{x}_{k-1|k-1} + K_k(y_k - H_k F_{k-1}\hat{x}_{k-1|k-1}) \\
&= (F_{k-1} - K_k H_k F_{k-1})\hat{x}_{k-1|k-1} + K_k y_k,
\end{aligned}
$$

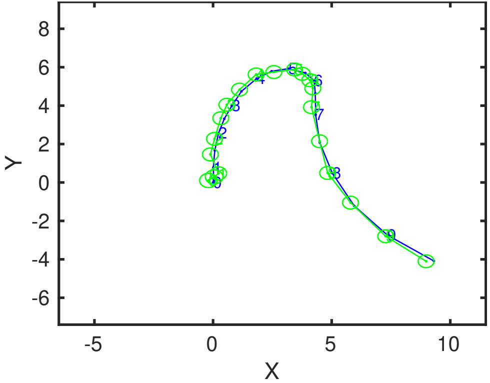
Predictor form

$$
\begin{aligned}
\hat{x}_{k+1|k} &= F_k \hat{x}_{k|k-1} + F_k K_k (y_k - H_k \hat{x}_{k|k-1}) \\
&= (F_k - F_k K_k H_k)\hat{x}_{k|k-1} + F_k K_k y_k
\end{aligned}
$$

Simulation Example (1/2)

Create a constant velocity model, simulate and Kalman filter.

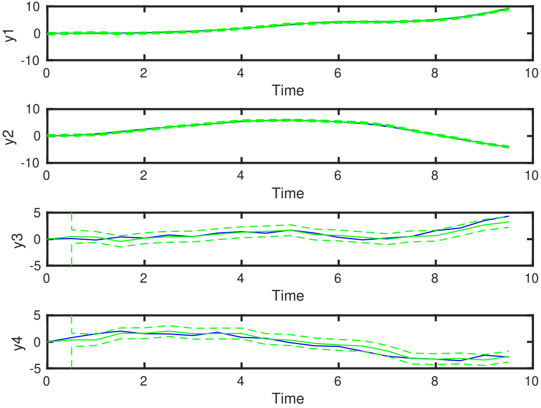
```
T = 0.5;
F = [1 0 T 0; 0 1 0 T; 0 0 1 0; 0 0 0 1];
G = [T^2/2 0; 0 T^2/2; T 0; 0 T];
H = [1 0 0 0; 0 1 0 0];
R = 0.03*eye(2);
m = lss(F, [], H,[], G*G', R, 1/T);
m.xlabel = {'X', 'Y', 'vX', 'vY'};
m.ylabel = {'X', 'Y'};
m.name = 'Constant␣velocity␣motion␣model';
z = simulate(m, 20);
xhat1 = kalman(m, z, 'alg', 2, 'k', 1); % Time-varying
xplot2(z, xhat1, 'conf', 90, [1 2]);
```



Simulation Example (2/2)

Covariance illustrated as confidence ellipsoids in 2D plots or confidence bands in 1D plots.

```
xplot(z, xhat1, 'conf', 99)
```



Tuning the KF

- The SNR ratio $\|Q\|/\|R\|$ is the most crucial, it sets the filter speeds. Note difference of real system and model used in the KF.
- Recommentation: fix R according to sensor specification/performance, and tune Q (motion models are anyway subjective approximations of reality).
- High SNR in the model, gives fast filter that is quick in adapting to changes/maneuvers, but with larger uncertainty (small bias, large variance).
- Conversely, low SNR in the model, gives slow filter that is slow in adapting to changes/maneuvers, but with small uncertainty (large bias, small variance).
- P_0 reflects the belief in the prior $x_1 \sim \mathcal{N}(\hat{x}_{1|0}, P_0)$. Possible to choose P_0 very large (and $\hat{x}_{1|0}$ arbitrary), if no prior information exists.
- Tune covariances in large steps (order of magnitudes)!

Optimality Properties

- For a linear model, the KF provides the WLS solution.
- The KF is the best linear unbiased estimator (BLUE).
- It is the Bayes optimal filter for linear model when x_0, v_k, e_k are Gaussian variables,

$$\begin{aligned} x_{k+1}|y_{1:k} &\sim \mathcal{N}(\hat{x}_{k+1|k}, P_{k+1|k}) \\ x_k|y_{1:k} &\sim \mathcal{N}(\hat{x}_{k|k}, P_{k|k}) \\ \varepsilon_k &\sim \mathcal{N}(0, S_k). \end{aligned}$$

Robustness and Sensitivity

The following concepts are relevant for all filtering applications, but they are most explicit for KF:

- **Observability** is revealed indirectly by $P_{k|k}$; monitor its rank or better condition number.
- **Divergence tests** Monitor performance measures and restart the filter after divergence.
- **Outlier rejection** monitor sensor observations.
- **Bias error** incorrect model gives bias in estimates.
- **Sensitivity analysis** uncertain model contributes to the total covariance.
- **Numerical issues** may give complex estimates.

Observability

1. Snapshot observability if H_k has full rank. WLS can be applied to estimate x.
2. Classical observability for time-invariant and time/varying case,

$$\mathcal{O} = \begin{pmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{pmatrix} \qquad \mathcal{O}_k = \begin{pmatrix} H_{k-n+1} \\ H_{k-n+2}F_{k-n+1} \\ H_{k-n+3}F_{k-n+2}F_{k-n+1} \\ \vdots \\ H_k F_{k-1} \dots F_{k-n+1} \end{pmatrix}.$$

3. The covariance matrix $P_{k|k}$ extends the observability condition by weighting with the measurement noise and to forget old information according to the process noise. Thus, (the condition number of) $P_{k|k}$ is the natural indicator of observability!

Divergence Tests

When is $\varepsilon_k \varepsilon_k^T$ significantly larger than its computed expected value $S_k = \mathsf{E}(\varepsilon_k \varepsilon_k^T)$ (note that $\varepsilon_k \sim \mathcal{N}(0, S_k)$)?

Principal reasons:

- Model errors
- Sensor model errors: offsets, drifts, incorrect covariances, scaling factor in all covariances
- Sensor errors: outliers, missing data
- Numerical issues

Solutions:

- In the first two cases, the filter has to be redesigned.
- In the last two cases, the filter has to be restarted.

Outlier Rejection

Outlier rejection as a hypothesis test

Let $H_0 : \varepsilon_k \sim \mathcal{N}(0, S_k)$, then

$$T(y_k) = \varepsilon_k^T S_k^{-1} \varepsilon_k \sim \chi^2_{n_{y_k}}$$

if everything works fine, and there is no outlier. If $T(y_k) > h_{P_{\text{FA}}}$, this is an indication of outlier, and the measurement update can be omitted.

In the case of several sensors, each sensor i should be monitored for outliers

$$T(y_k^i) = (\varepsilon_k^i)^T S_k^{-1} \varepsilon_k^i \sim \chi^2_{n_{y_k^i}}.$$

Sensitivity analysis: parameter uncertainty

Sensitivity analysis can be done with respect to uncertain parameters with known covariance matrix using for instance Gauss approximation formula.

- Assume $F(\theta), G(\theta), H(\theta), Q(\theta), R(\theta)$ have uncertain parameters θ with $\mathsf{E}(\theta) = \hat{\theta}$ and $\text{cov}(\theta) = P_\theta$.
- The state estimate $\hat{x}_k$ is a stochastic variable with four stochastic sources, v_k, e_k, x_1 at one hand, and θ on the other hand.
- The law of total variance ($\text{var}(X) = \mathsf{E}\,\text{var}(X|Y) + \text{var}\,\mathsf{E}(X|Y)$) and Gauss approximation formula ($\text{var}(h(Y)) \approx h'_Y(\bar{Y})\,\text{var}(Y)(h'_Y(\bar{Y}))^T$) gives

$$\text{cov}(\hat{x}_{k|k}) \approx P_{k|k} + \frac{d\hat{x}_{k|k}}{d\theta} P_\theta \left(\frac{d\hat{x}_{k|k}}{d\theta}\right)^T.$$

- The gradient $d\hat{x}_{k|k}/d\theta$ can be computed numerically by simulations.

Numerical Issues

Some simple fixes if problem occurs:

- Assure that the covariance matrix is symmetric
  `P = 0.5*P + 0.5*P'.`
- Use the more numerically stable Joseph's form for the measurement update of the covariance matrix:

$$P_{k|k} = (I - K_k H_k) P_{k|k-1} (I - K_k H_k)^T + K_k R_k K_k^T.$$

- Assure that the covariance matrix is positive definite by setting negative eigenvalues in P to zero or small positive values.
- Avoid singular $R = 0$, even for constraints.
- Dithering. Increase Q and R if needed; this can account for all kind of model errors.

Kalman Filter Approximations (EKF, UKF)

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Chapter 8 Overview

- Nonlinear transformations.
- Details of the EKF algorithms.
- Numerical methods to compute Jacobian and Hessian in the Taylor expansion.
- An alternative EKF version without the Ricatti equation.
- The unscented Kalman filter (UKF).

EKF1 and EKF2 principle

Apply TT1 and TT2, respectively, to the dynamic and observation models. For instance,

$$x_{k+1} = f(x_k) + v_k = f(\hat{x}) + g'(\hat{x})(x - \hat{x}) + \frac{1}{2}(x - \hat{x})^T g''(\xi)(x - \hat{x}).$$

- EKF1 neglects the rest term.
- EKF2 compensates with the mean and covariance of the rest term using $\xi = \hat{x}$.

EKF1 Algorithm

$$S_k = h'_x(\hat{x}_{k|k-1})P_{k|k-1}(h'_x(\hat{x}_{k|k-1}))^T + h'_e(\hat{x}_{k|k-1})R_k(h'_e(\hat{x}_{k|k-1}))^T$$

$$\begin{aligned}
K_k &= P_{k|k-1}(h'_x(\hat{x}_{k|k-1}))^T S_k^{-1} \\
\varepsilon_k &= y_k - h(\hat{x}_{k|k-1}, 0) \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k \varepsilon_k \\
P_{k|k} &= P_{k|k-1} - P_{k|k-1}(h'_x(\hat{x}_{k|k-1}))^T S_k^{-1} h'_x(\hat{x}_{k|k-1}) P_{k|k-1}
\end{aligned}$$

$$\begin{aligned}
\hat{x}_{k+1|k} &= f(\hat{x}_{k|k}, 0) \\
P_{k+1|k} &= f'_x(\hat{x}_{k|k})P_{k|k}(f'_x(\hat{x}_{k|k}))^T + f'_v(\hat{x}_{k|k})Q_k(f'_v(\hat{x}_{k|k}))^T
\end{aligned}$$

EKF1 and EKF2 Algorithm

$$
\begin{aligned}
S_k &= h'_x(\hat{x}_{k|k-1})P_{k|k-1}(h'_x(\hat{x}_{k|k-1}))^T + h'_e(\hat{x}_{k|k-1})R_k(h'_e(\hat{x}_{k|k-1}))^T \\
&\quad + \tfrac{1}{2}\left[\mathrm{tr}(h''_{i,x}(\hat{x}_{k|k-1})P_{k|k-1}h''_{j,x}(\hat{x}_{k|k-1})P_{k|k-1})\right]_{ij} \\
K_k &= P_{k|k-1}(h'_x(\hat{x}_{k|k-1}))^T S_k^{-1} \\
\varepsilon_k &= y_k - h(\hat{x}_{k|k-1}, 0) - \tfrac{1}{2}\left[\mathrm{tr}(h''_{i,x}P_{k|k-1})\right]_i \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k\varepsilon_k \\
P_{k|k} &= P_{k|k-1} - P_{k|k-1}(h'_x(\hat{x}_{k|k-1}))^T S_k^{-1} h'_x(\hat{x}_{k|k-1})P_{k|k-1} \\
&\quad + \tfrac{1}{2}\left[\mathrm{tr}(h''_{i,x}(\hat{x}_{k|k-1})P_{k|k-1}h''_{j,x}(\hat{x}_{k|k-1})P_{k|k-1})\right]_{ij} \\
\hat{x}_{k+1|k} &= f(\hat{x}_{k|k}, 0) + \tfrac{1}{2}\left[\mathrm{tr}(f''_{i,x}P_{k|k})\right]_i \\
P_{k+1|k} &= f'_x(\hat{x}_{k|k})P_{k|k}(f'_x(\hat{x}_{k|k}))^T + f'_v(\hat{x}_{k|k})Q_k(f'_v(\hat{x}_{k|k}))^T \\
&\quad + \tfrac{1}{2}\left[\mathrm{tr}(f''_{i,x}(\hat{x}_{k|k})P_{k|k}f''_{j,x}(\hat{x}_{k|k})P_{k|k})\right]_{ij}
\end{aligned}
$$

NB!

This form of the EKF2 (as given in the book) is disregarding second order terms of the process noise! See, *e.g.*, my thesis for the full expressions.

Comments

- The EKF1, using the TT1 transformation, is obtained by letting both Hessians f''_x and h''_x be zero.
- Analytic Jacobian and Hessian needed. If not available, use numerical approximations (done in Signal and Systems Lab by default!)
- The complexity of EKF1 is as in KF n_x^3 due to the FPF^T operation.
- The complexity of EKF2 is n_x^5 due to the $F_i P F_j^T$ operation for $i, j = 1, \ldots, n_x$.
- Dithering is good! That is, increase Q and R from the simulated values to account for the approximation errors.

EKF Variations

- The standard EKF linearizes around the current state estimate.
- The *linearized Kalman filter* linearizes around some reference trajectory.
- The *error state Kalman filter*, also known as the *complementary Kalman filter*, estimates the state error $\tilde{x}_k = x_k - \hat{x}_k$ with respect to some approximate or reference trajectory. Feedforward or feedback configurations.

linearized Kalman filter = feedforward error state Kalman filter
EKF = feedback error state Kalman filter

Derivative-Free Algorithms

Numeric derivatives are preferred in the following cases:

- The nonlinear function is too complex.
- The derivatives are too complex functions.
- A user-friendly algorithm is desired, with as few user inputs as possible.

This can be achieved with either numerical approximation or using sigma points!

Nonlinear transformations (NLT)

Consider a second order Taylor expansion of a function $z = g(x)$:

$$z = g(x) = g(\hat{x}) + g'(\hat{x})(x - \hat{x}) + \underbrace{\tfrac{1}{2}(x - \hat{x})^T g''(\xi)(x - \hat{x})}_{r(x;\hat{x},g''(\xi))}.$$

The rest term is negligible and EKF works fine if:

- the model is almost linear
- or the SNR is high, so $\|x - \hat{x}\|$ can be considered small.

The size of the rest term can be approximated *a priori*.

Note: the size may depend on the choice of state coordinates!

If the rest term is large, use either of

- the second order compensated EKF that compensates for the mean and covariance of $r(x;\hat{x},g''(\xi)) \approx r(x;\hat{x},g''(\hat{x}))$.
- the unscented KF (UKF).

TT1: first order Taylor approximation

The first order Taylor term gives a contribution to the covariance:

$$x \sim \mathcal{N}\big(\hat{x}, P\big) \rightarrow \mathcal{N}\big(g(\hat{x}), [g_i'(\hat{x})P(g_j'(\hat{x}))^T]_{ij}\big) = \mathcal{N}\big(g(\hat{x}), g'(\hat{x})P(g'(\hat{x}))^T\big)$$

- This is sometimes called Gauss' approximation formula.
- Here $[A]_{ij}$ means element i, j in the matrix A. This is used in EKF1 (EKF with first order Taylor expansion). Leads to a KF where nonlinear functions are approximated with their Jacobians.
- Compare with the linear transformation rule

$$z = Gx, \qquad x \sim \mathcal{N}\big(\hat{x}, P\big) \qquad \longrightarrow \quad z \sim \mathcal{N}\big(G\hat{x}, GPG^T\big).$$

- Note that GPG^T can be written $[G_i P G_j^T]_{ij}$, where G_i denotes row i of G.

TT2: second order Taylor approximation

The second order Taylor term contributes both to the mean and covariance as follows:

$$x \sim \mathcal{N}\big(\hat{x}, P\big) \to \mathcal{N}\big(g(\hat{x}) + \tfrac{1}{2}[\mathrm{tr}(g_i''(\hat{x})P)]_i,\ [g_i'(\hat{x})P(g_j'(\hat{x}))^T + \tfrac{1}{2}\,\mathrm{tr}(Pg_i''(\hat{x})Pg_j''(\hat{x}))]_{ij}\big)$$

- This is used in EKF2 (EKF with second order Taylor expansion). Leads to a KF where nonlinear functions are approximated with their Jacobians and Hessians.
- UKF tries to do this approximation numerically, *without* forming the Hessian $g''(x)$ explicitly. This reduces the n_x^5 complexity in $\big[\mathrm{tr}\big(Pg_i''(\hat{x})Pg_j''(\hat{x})\big)\big]_{ij}$ to n_x^3 complexity.

MC: Monte Carlo sampling

Generate N samples, transform them, and fit a Gaussian distribution

$$
\begin{aligned}
x^{(i)} &\sim \mathcal{N}\big(\hat{x}, P\big) \\
z^{(i)} &= g(x^{(i)}) \\
\mu_z &= \frac{1}{N}\sum_{i=1}^{N} z^{(i)} \\
P_z &= \frac{1}{N-1}\sum_{i=1}^{N}\big(z^{(i)} - \mu_z\big)\big(z^{(i)} - \mu_z\big)^T
\end{aligned}
$$

Not commonly used in nonlinear filtering, but a valid and solid approach!

UT: the unscented transform

At first sight, similar to MC:

Generate $2n_x+1$ *sigma points*, transform these, and fit a Gaussian distribution:

$$
\begin{aligned}
x^{(0)} &= \hat{x} \\
x^{(\pm i)} &= \hat{x} \pm \sqrt{n_x+\lambda}P^{1/2}_{:,i}, \quad i = 1,2,\dots,n_x \\
z^{(i)} &= g(x^{(i)}) \\
\mathsf{E}(z) &\approx \frac{\lambda}{2(n_x+\lambda)} z^{(0)} + \sum_{i=-n_x}^{n_x} \frac{1}{2(n_x+\lambda)} z^{(i)} \\
\mathsf{cov}(z) &\approx \Big(\frac{\lambda}{2(n_x+\lambda)} + (1-\alpha^2+\beta)\Big)\big(z^{(0)}-\mathsf{E}(z)\big)\big(z^{(0)}-\mathsf{E}(z)\big)^T \\
&\quad + \sum_{i=-n_x}^{n_x} \frac{1}{2(n_x+\lambda)}\big(z^{(i)}-\mathsf{E}(z)\big)\big(z^{(i)}-\mathsf{E}(z)\big)^T
\end{aligned}
$$

UT: design parameters

- λ is defined by $\lambda = \alpha^2(n_x + \kappa) - n_x$.
- α controls the spread of the sigma points and is suggested to be chosen around 10^{-3}.
- β compensates for the distribution, and should be chosen to $\beta = 2$ for Gaussian distributions.
- κ is usually chosen to zero.

Note

- $n_x + \lambda = \alpha^2 n_x$ when $\kappa = 0$.
- The weights sum to one for the mean, but sum to $2 - \alpha^2 + \beta \approx 4$ for the covariance. Note also that the weights are not in $[0, 1]$.
- The mean has a large negative weight!
- If $n_x + \lambda \to 0$, then UT and TT2 (and hence UKF and EKF2) are identical for $n_x = 1$, otherwise closely related!

Example 1: squared norm

Squared norm of a Gaussian vector has a known distribution:

$$z = g(x) = x^T x, \quad x \sim \mathcal{N}(0, I_n) \Rightarrow z \sim \chi^2(n).$$

Theoretical distribution is $\chi^2(n)$ with mean n and variance $2n$. The mean and variance are below summarized as a Gaussian distribution. Using 10 000 Monte Carlo simulations.

n	**TT1**	**TT2**	**UT1**	**UT2**	**MCT**
1	$\mathcal{N}(0,0)$	$\mathcal{N}(1,2)$	$\mathcal{N}(1,2)$	$\mathcal{N}(1,2)$	$\mathcal{N}(1.02, 2.15)$
2	$\mathcal{N}(0,0)$	$\mathcal{N}(2,4)$	$\mathcal{N}(2,2)$	$\mathcal{N}(2,8)$	$\mathcal{N}(2.02, 4.09)$
3	$\mathcal{N}(0,0)$	$\mathcal{N}(3,6)$	$\mathcal{N}(3,0)$	$\mathcal{N}(3,18)$	$\mathcal{N}(3.03, 6.3)$
4	$\mathcal{N}(0,0)$	$\mathcal{N}(4,8)$	$\mathcal{N}(4,-4)$	$\mathcal{N}(4,32)$	$\mathcal{N}(4.03, 8.35)$
5	$\mathcal{N}(0,0)$	$\mathcal{N}(5,10)$	$\mathcal{N}(5,-10)$	$\mathcal{N}(5,50)$	$\mathcal{N}(5.08, 10.4)$
Theory	$\mathcal{N}(0,0)$	$\mathcal{N}(n,2n)$	$\mathcal{N}(n,(3-n)n)$	$\mathcal{N}(n,2n^2)$	$\to \mathcal{N}(n,2n)$

Conclusion: TT2 works, not the unscented transforms.

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1	$\mathcal{N}(0,0)$	$\mathcal{N}(1,2)$	$\mathcal{N}(1,2)$	$\mathcal{N}(1,2)$	$\mathcal{N}(1.02,2.15)$
2	$\mathcal{N}(0,0)$	$\mathcal{N}(2,4)$	$\mathcal{N}(2,2)$	$\mathcal{N}(2,8)$	$\mathcal{N}(2.02,4.09)$
3	$\mathcal{N}(0,0)$	$\mathcal{N}(3,6)$	$\mathcal{N}(3,0)$	$\mathcal{N}(3,18)$	$\mathcal{N}(3.03,6.3)$
4	$\mathcal{N}(0,0)$	$\mathcal{N}(4,8)$	$\mathcal{N}(4,-4)$	$\mathcal{N}(4,32)$	$\mathcal{N}(4.03,8.35)$
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Theory	$\mathcal{N}(0,0)$	$\mathcal{N}(n,2n)$	$\mathcal{N}(n,(3-n)n)$	$\mathcal{N}(n,2n^2)$	$\to \mathcal{N}(n,2n)$

Conclusion: TT2 works, not the unscented transforms.

Example 2: radar

Conversion of polar measurements to Cartesian position:

$$z = g(x) = \begin{pmatrix} x_1 \cos(x_2) \\ x_1 \sin(x_2) \end{pmatrix}$$

X	**TT1**	**TT2**
$\begin{pmatrix} 3.0 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}$	$\begin{pmatrix} 3.0 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 9.0 \end{pmatrix}$	$\begin{pmatrix} 2.0 \\ -0.0 \end{pmatrix}, \begin{pmatrix} 3.0 & 0.0 \\ 0.0 & 10.0 \end{pmatrix}$
$\begin{pmatrix} 3.0 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}$	$\begin{pmatrix} 2.6 \\ 1.5 \end{pmatrix}, \begin{pmatrix} 3.0 & -3.5 \\ -3.5 & 7.0 \end{pmatrix}$	$\begin{pmatrix} -1.4 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 27.0 & 2.5 \\ 2.5 & 9.0 \end{pmatrix}$
$\begin{pmatrix} 3.0 \\ 0.8 \end{pmatrix}, \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}$	$\begin{pmatrix} 2.1 \\ 2.1 \end{pmatrix}, \begin{pmatrix} 5.0 & -4.0 \\ -4.0 & 5.0 \end{pmatrix}$	$\begin{pmatrix} 2.1 \\ 2.1 \end{pmatrix}, \begin{pmatrix} 9.0 & 0.0 \\ 0.0 & 13.0 \end{pmatrix}$

UT1	**UT2**	**MCT**
$\begin{pmatrix} 1.8 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 3.7 & 0.0 \\ 0.0 & 2.9 \end{pmatrix}$	$\begin{pmatrix} 1.5 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 5.5 & 0.0 \\ 0.0 & 9.0 \end{pmatrix}$	$\begin{pmatrix} 1.8 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 2.5 & 0.0 \\ 0.0 & 4.4 \end{pmatrix}$
$\begin{pmatrix} 1.6 \\ 0.9 \end{pmatrix}, \begin{pmatrix} 3.5 & 0.3 \\ 0.3 & 3.1 \end{pmatrix}$	$\begin{pmatrix} 1.3 \\ 0.8 \end{pmatrix}, \begin{pmatrix} 6.4 & -1.5 \\ -1.5 & 8.1 \end{pmatrix}$	$\begin{pmatrix} 1.6 \\ 0.9 \end{pmatrix}, \begin{pmatrix} 2.9 & -0.8 \\ -0.8 & 3.9 \end{pmatrix}$
$\begin{pmatrix} 1.3 \\ 1.3 \end{pmatrix}, \begin{pmatrix} 3.3 & 0.4 \\ 0.4 & 3.3 \end{pmatrix}$	$\begin{pmatrix} 1.1 \\ 1.1 \end{pmatrix}, \begin{pmatrix} 7.2 & -1.7 \\ -1.7 & 7.2 \end{pmatrix}$	$\begin{pmatrix} 1.3 \\ 1.3 \end{pmatrix}, \begin{pmatrix} 3.4 & -1.0 \\ -1.0 & 3.4 \end{pmatrix}$

Example 3: standard sensor networks measurements

Standard measurements:

$$g_{\text{TOA}}(x) = \|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$
$$g_{\text{DOA}}(x) = \text{arctan2}(x_1, x_2),$$

TOA 2D: $g(x) = \|x\|$	
X	$\mathcal{N}([3;0],[1,0;0,10])$
TT1	$\mathcal{N}(3,1)$
TT2	$\mathcal{N}(4.67,6.56)$
UT2	$\mathcal{N}(4.08,3.34)$
MCT	$\mathcal{N}(4.08,1.94)$

DOA: $g(x) = \text{arctan2}(x_2, x_1)$	
X	$\mathcal{N}([3;0],[10,0;0,1])$
TT1	$\mathcal{N}(0,0.111)$
TT2	$\mathcal{N}(0,0.235)$
UT2	$\mathcal{N}(0.524,1.46)$
MCT	$\mathcal{N}(0.0702,1.6)$

Conclusion: UT works slightly better than TT1 and TT2. Studying RSS measurements,

$$g_{\text{RSS}}(x) = c_0 - c_2 \cdot 10\log_{10}(|x\|^2),$$

gives similar results.

KF, EKF and UKF in one framework

Lemma 7.1 If

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} P_{xx} & P_{xy} \\ P_{xy} & P_{yy} \end{pmatrix}\right) = \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, P\right)$$

Then, the conditional distribution for X, given the observed $Y = y$, is Gaussian distributed:

$$(X|Y = y) \sim \mathcal{N}(\mu_x + P_{xy}P_{yy}^{-1}(y - \mu_y), P_{xx} - P_{xy}P_{yy}^{-1}P_{yx})$$

Connection to the Kalman filter

The Kalman gain is in this notation given by

$$K_k = P_{xy}P_{yy}^{-1}.$$

Kalman Filter Algorithm (1/2)

Time update: Let

$$\bar{x} = \begin{pmatrix} x_k \\ v_k \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \hat{x}_{k|k} \\ 0 \end{pmatrix}, \begin{pmatrix} P_{k|k} & 0 \\ 0 & Q_k \end{pmatrix}\right)$$
$$z = x_{k+1} = f(x_k, u_k, v_k) = g(\bar{x}).$$

The transformation approximation (UT, MC, TT1, TT2) gives

$$z \sim \mathcal{N}(\hat{x}_{k+1|k}, P_{k+1|k}).$$

Kalman Filter Algorithm (2/2)

Measurement update: Let

$$\bar{x} = \begin{pmatrix} x_k \\ e_k \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \hat{x}_{k|k-1} \\ 0 \end{pmatrix}, \begin{pmatrix} P_{k|k-1} & 0 \\ 0 & R_k \end{pmatrix}\right)$$

$$z = \begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} x_k \\ h(x_k, u_k, e_k) \end{pmatrix} = g(\bar{x}).$$

The transformation approximation (UT, MC, TT1, TT2) gives

$$z \sim \mathcal{N}\left(\begin{pmatrix} \hat{x}_{k|k-1} \\ \hat{y}_{k|k-1} \end{pmatrix}, \begin{pmatrix} P^{xx}_{k|k-1} & P^{xy}_{k|k-1} \\ P^{yx}_{k|k-1} & P^{yy}_{k|k-1} \end{pmatrix}\right).$$

The measurement update is now

$$K_k = P^{xy}_{k|k-1}\big(P^{yy}_{k|k-1}\big)^{-1},$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k\big(y_k - \hat{y}_{k|k-1}\big),$$

$$P_{k|k} = P_{k|k-1} - K_k P^{yx}_{k|k-1}.$$

Comments

- The filter obtained using TT1 is equivalent to the standard EKF1.
- The filter obtained using TT2 is equivalent to EKF2.
- The filter obtained using UT is equivalent to UKF.
- The Monte Carlo approach should be the most accurate, since it asymptotically computes the correct first and second order moments.
- There is a freedom to mix transform approximations in the time and measurement update.

Choice of Nonlinear Filter

- Depends mainly on:
  (i) SNR.
  (ii) the degree of nonlinearity.
  (iii) the degree of non-Gaussian noise, in particular if any distribution is multi-modal (has several local maxima).
- SNR and degree of nonlinearity is connected through the rest term, whose expected value is:

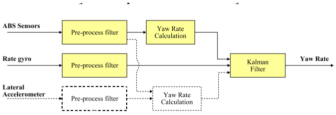
$$\mathsf{E}(x-\hat{x})^T g''(\xi)(x-\hat{x}) = \mathsf{E}\Big(\operatorname{tr}\big(g''(\xi)(x-\hat{x})(x-\hat{x})^T\big)\Big) = \operatorname{tr}\big(g''(\xi)P\big)$$

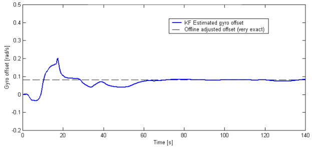
  Small rest term requires either high SNR (small P) or almost linear functions (small f'' and h'').
- If the rest term is small, use EKF1.
- If the rest term is large, and the nonlinearities are essentially quadratic (example $x^T x$) use EKF2.
- If the rest term is large, and the nonlinearities are *not* essentially quadratic try UKF.
- If the functions are severly nonlinear or any distribution is multi-modal, consider filterbanks or particle filter.

Virtual Yaw Rate Sensor

- Yaw rate subject to bias, orientation error increases linearly in time.
- Wheel speeds also give a gyro, where the orientation error grows linearly in distance.

Model, with state vector $x_k = (\dot{\psi}_k, \ddot{\psi}_k, b_k, \frac{r_{k,3}}{r_{k,4}})$ and the measurements





$$y_k^2 = \frac{\omega_3 r_{\text{nom}} + \omega_4 r_{\text{nom}}}{2} \frac{2}{B} \frac{\frac{\omega_3}{\omega_4}\frac{r_{k,3}}{r_{k,4}} - 1}{\frac{\omega_3}{\omega_4}\frac{r_{k,3}}{r_{k,4}} + 1} + e_k^2.$$

http://youtu.be/d9rzCCIBS9I

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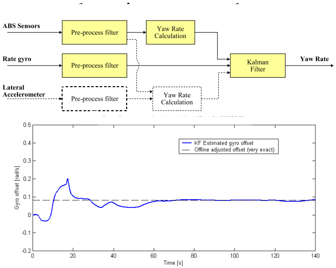
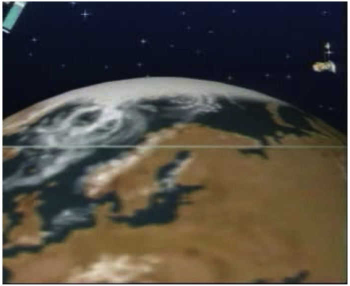


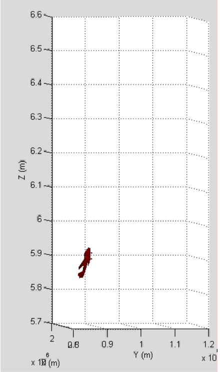
http://youtu.be/d9rzCCIBS9I

Sounding Rocket

Navigation grade IMU gives accurate dead-reckoning, but drift may cause return at bad places.
GPS is restricted for high speeds and high accelerations.
Fusion of IMU and GPS when pseudo-ranges are available, with IMU support to tracking loops inside GPS.

- Loose integration: direct fusion approach $y_k = p_k + e_k$.
- Tight integration: TDOA fusion approach $y_k^i = |p_k - p_k^i|/c + t_k + e_k$.





http://youtu.be/zRHFXfZLQ64

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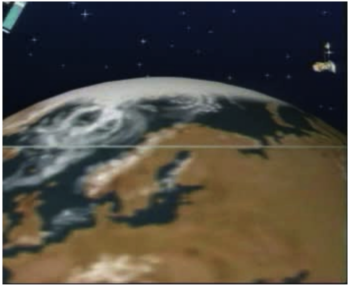

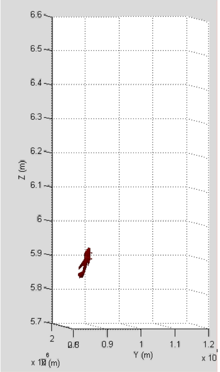

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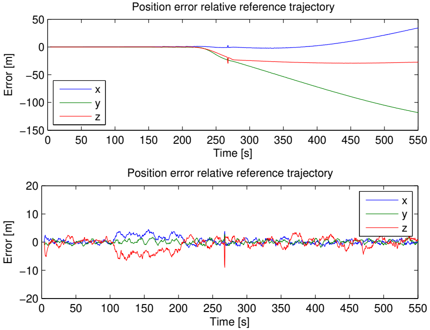

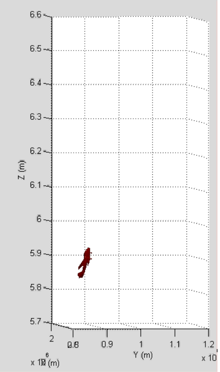

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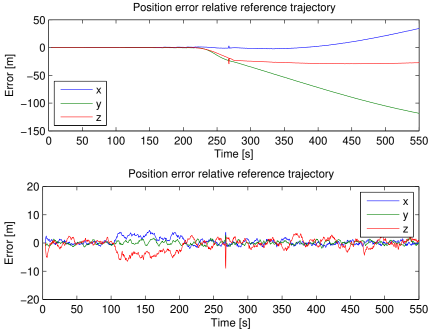

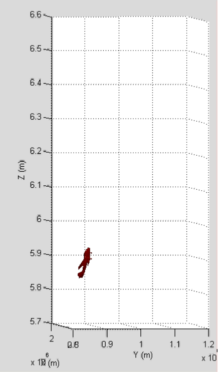

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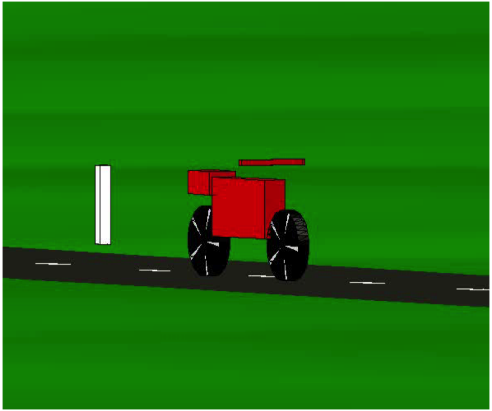
MC Leaning Angle

- Headlight steering, ABS and anti-spin systems require leaning angle.
- Gyro very expensive for this application.
- Combination of accelerometers investigated, lateral and downward acc worked fine in EKF.



http://youtu.be/hT6S1FgHxOcO

Model, where z_y, z_z, a_1, a_2, J are constants relating to geometry and inertias of the motorcycle, $u = v_x$

$$x = \begin{pmatrix} \varphi & \dot{\varphi} & \ddot{\varphi} & \dot{\psi} & \ddot{\psi} & \delta_{ay} & \delta_{az} & \delta_{\dot{\varphi}} \end{pmatrix}^T.$$

$$y = h(x) = \begin{pmatrix} a_y \\ a_z \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} u x_4 - z_y x_3 + z_y x_4^2 \tan(x_1) + g \sin(x_1) + x_6 \\ -u x_4 \tan(x_1) - z_z \left(x_2^2 + x_4^2 \tan^2(x_1) \right) + g \cos(x_1) + x_7 \\ -a_1 x_3 + a_2 x_4^2 \tan(x_1) - u x_4 J + x_6 \end{pmatrix}$$

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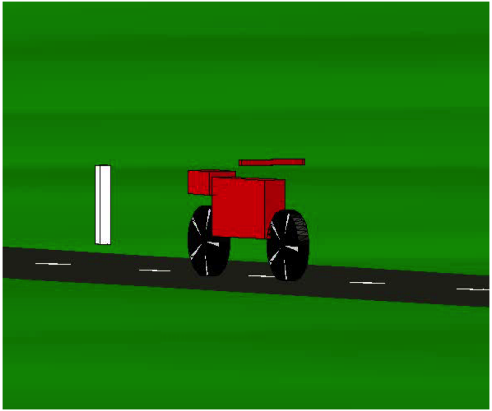

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Summary

Summary Lecture 6

Key tool for a unified derivation of KF, EKF, UKF.

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} P_{xx} & P_{xy} \\ P_{xy} & P_{yy} \end{pmatrix}\right)$$
$$\Rightarrow (X|Y=y) \sim \mathcal{N}(\mu_x + P_{xy}P_{yy}^{-1}(y-\mu_y), P_{xx} - P_{xy}P_{yy}^{-1}P_{yx})$$

The Kalman gain is in this notation given by $K_k = P_{xy}P_{yy}^{-1}$.

- In KF, P_{xy} and P_{yy} follow from a linear Gaussian model.
- In EKF, P_{xy} and P_{yy} can be computed from a linearized model (requires analytic gradients).
- In EKF and UKF, P_{xy} and P_{yy} computed by NLT for transformation of $(x^T, v^T)^T$ and $(x^T, e^T)^T$, respectively. No gradients required, just function evaluations.

Gustaf Hendeby

gustaf.hendeby@liu.se

www.liu.se

LINKÖPING UNIVERSITY